



A COHEN TYPE INEQUALITY FOR LAGUERRE-SOBOLEV EXPANSIONS

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ABSTRACT. Let introduce the Sobolev-type inner product

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)d\mu(x) + Mf(0)g(0) + Nf'(0)g'(0),$$

where

$$d\mu(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x} dx, \quad M, N \geq 0, \quad \alpha > -1.$$

In this paper we prove a Cohen type inequality for the Fourier expansion in terms of the orthonormal polynomials associated with the above Sobolev inner product. In particular, for $M = N = 0$, we extend the result of Markett ([14])

1. INTRODUCTION

Polynomials orthogonal with respect to an inner product

$$\langle f, g \rangle = \int_E f(x)g(x)d\mu(x) + Mf(a)g(a) + Nf'(a)g'(a),$$

where a is a real number and $d\mu$ is a positive Borel measure supported on an infinite subset E of the real line have been considered by several authors (see [1] and the references therein). Such polynomials are known in the literature as Sobolev type orthogonal polynomials. A special emphasis was done to algebraic and analytic properties of such polynomials, in particular, the distribution of their zeros taking into account the location of the point a with respect to the set E .

When E is the interval $[0, +\infty)$, $a = 0$, and $d\mu$ is the Laguerre measure, Koekoek [10] analyzed some analytic properties of the zeros of the so called Laguerre Sobolev orthogonal polynomials as well as the hypergeometric character of such polynomials. Later on, Koekoek and Meijer [11] have studied the same problems in a more general framework, when the inner product involves the Laguerre weight and derivatives of higher order at $a = 0$. An interesting problem is related to the asymptotic

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behaviour of such polynomials. A first approach was done by Alvarez Nodarse and Moreno Balcazar in [2]. Outer and inner strong asymptotics, relative asymptotics as well as the analog of the Mehler-Heine for such polynomials are studied. The main tool is the representation of Laguerre Sobolev-type in terms of three standard Laguerre polynomials. The behaviour of the coefficients in this expression allows the analysis of such asymptotic properties.

A further work concerning the Fourier expansions with respect to Sobolev type orthogonal polynomials was done by one of the authors and his coworkers (see [12], [15]) when E is a bounded interval, the measure $d\mu$ belongs to the Nevai class, and a is a point of the real line. Thus it seems to be natural to analyze the Fourier expansions in terms of Laguerre Sobolev type orthogonal polynomials as well as to compare it with the Fourier expansions in terms of standard Laguerre orthogonal polynomials.

In such a direction, the aim of this paper is to derive a lower bound for the norm of the partial sums of the Fourier expansions in terms of Laguerre Sobolev type orthonormal polynomials, the well known Cohen type inequality in the framework of Approximation Theory. A Cohen type inequality have been established in other contexts, e.g., on compact groups or for classical orthogonal expansions. We refer to [5], [6], [7], [8], [9], [14], and references therein.

The structure of the manuscript is the following. In Section 2 the basic background about the properties of Laguerre orthogonal polynomials to be used in the sequel is given. Section 3 deals with some weight estimates for Laguerre Sobolev type orthonormal polynomials using p -norms. Section 4 contains the main result of the manuscript (Theorem 1). Here a lower bound for the p -norm of the partial sums of some balanced Fourier expansions in terms of Laguerre Sobolev type orthonormal polynomials is deduced. As a simple consequence (Corollary 2) the divergence of such partial sums when p is located outside the Pollard interval is deduced.

2. THE CLASSICAL LAGUERRE POLYNOMIALS

For $\alpha > -1$ we denote by $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ the sequence of Laguerre polynomials, orthogonal on $[0, +\infty)$ with respect to the measure $d\mu(x) = \frac{1}{\Gamma(\alpha+1)}x^\alpha e^{-x}dx$, (see [17, Chapter V]). They are normalized in such a way that $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} =$

$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \sim n^\alpha$. Here the notation $u_n \sim v_n$ means that there exist positive real numbers c_1 and c_2 such that $c_1 u_n \leq v_n \leq c_2 u_n$ for n large enough while $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1.

We denote the orthonormal Laguerre polynomial of degree n by

$$(1) \quad \hat{L}_n^{(\alpha)}(x) = (h_n^{(\alpha)})^{-1/2} L_n^{(\alpha)}(x)$$

where $h_n^{(\alpha)} = \int_0^\infty [L_n^{(\alpha)}(x)]^2 d\mu(x) \cong \frac{n^\alpha}{\Gamma(\alpha+1)}$.

Now we list some properties of the Laguerre polynomials which we will use in the sequel. They satisfy a structure relation (see [17, formula (5.1.14)])

$$(2) \quad x L_n^{(\alpha+1)}(x) = (n + \alpha + 1) L_n^{(\alpha)}(x) - (n + 1) L_{n+1}^{(\alpha)}(x)$$

as well as the following relation for the derivatives (see [17, formula (5.1.14)]):

$$(3) \quad \frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x).$$

On the other hand, the following integral formula for Laguerre polynomials holds (see [3, p. 291, formula (6.2.38)])

$$(4) \quad \int_0^\infty L_n^{(\alpha+i)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)},$$

where $i = 0, 1, \dots$.

The formula of Mehler-Heine type for Laguerre orthonormal polynomials is (see [17, Theorem 8.1.3])

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\hat{L}_n^{(\alpha)}(x/(n+j))}{n^{\alpha/2}} = \sqrt{\Gamma(\alpha+1)} x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

where J_α is the Bessel function. This formula holds uniformly on compact subsets of \mathbf{C} and uniformly for $j \in \mathbf{N} \cup \{0\}$.

The following result was given by Markett ([13, Lemma 1])

$$(6) \quad \left(\int_0^\infty |\hat{L}_n^{(\alpha+\beta)}(x) e^{-x/2} x^{\alpha/2}|^p dx \right)^{1/p} \sim \begin{cases} n^{1/p-1/2-\beta/2} (\log n)^{1/p} & \text{if } \beta = 2/p - 1/2, \quad 1 \leq p \leq 4, \\ n^{\beta/2-1/p} & \text{if } \beta > 2/p - 1/2, \quad 1 \leq p \leq 4, \\ n^{\beta/2-1/p} & \text{if } \beta > 4/3p - 1/3, \quad 4 < p \leq \infty. \end{cases}$$

In particular, we get

Lemma 1. For $\alpha \geq 0$ and $j = 0, 1, 2$

$$\left(\int_0^\infty |x^j \hat{L}_n^{(\alpha+2j)}(x) e^{-x/2}|^p x^\alpha dx \right)^{1/p} \sim \begin{cases} n^{-1/4} (\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha/2 - (\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p \leq \infty, \end{cases}$$

and

$$\left(\int_0^\infty |x^j \hat{L}_n^{(\alpha+2j)}(x) e^{-x/2} x^{\alpha/2}|^p dx \right)^{1/p} \sim \begin{cases} n^{-1/4} (\log n)^{1/p} & \text{if } p = 4, \\ n^{-1/p} & \text{if } 4 < p \leq \infty. \end{cases}$$

Proof. Concerning the first relation, we use (6) with α, β replaced by $2\alpha/p + 2j$ and $\alpha - 2\alpha/p$ respectively.

Concerning the second relation, we use (6) with α replaced by $\alpha + 2j$ and $\beta = 0$. \square

3. ESTIMATES FOR LAGUERRE-SOBOLEV TYPE POLYNOMIALS

Let introduce the Sobolev-type inner product

$$(7) \quad \langle f, g \rangle = \int_0^\infty f(x)g(x)d\mu(x) + Mf(0)g(0) + Nf'(0)g'(0),$$

where

$$d\mu(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x} dx, \quad M, N \geq 0, \quad \alpha > -1.$$

In [10] Koekoek and Meijer introduced a sequence of polynomials $\{L_n^{(\alpha, M, N)}(x)\}_{n=0}^\infty$ which are orthogonal with respect to the above inner product. These polynomials are known in the literature as Laguerre-Sobolev type polynomials.

The representation of the polynomials $L_n^{(\alpha, M, N)}$ in terms of the sequence of Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n \geq 0}$ is (see [2], [10])

$$(8) \quad L_n^{(\alpha, M, N)}(x) = B_0(n)L_n^{(\alpha)}(x) + B_1(n)xL_{n-1}^{(\alpha+2)}(x) + B_2(n)x^2L_{n-2}^{(\alpha+4)}(x)$$

where, as a convention, $L_i^{(\alpha)}(x) = 0$, for $i = -1, -2$ and

- a) if $M > 0$ and $N > 0$ then $B_0(n) \cong \frac{-Nn^{\alpha+3}}{(\alpha+1)\Gamma(\alpha+4)}$, $B_1(n) \cong \frac{-N(\alpha+2)n^{\alpha+2}}{(\alpha+1)\Gamma(\alpha+4)}$,
 $B_2(n) \cong \frac{MNn^{2\alpha+2}}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}$,
- b) if $M = 0$ and $N > 0$ then $B_0(n) \cong \frac{-Nn^{\alpha+3}}{(\alpha+1)\Gamma(\alpha+4)}$, $B_1(n) \cong \frac{-N(\alpha+2)n^{\alpha+2}}{(\alpha+1)\Gamma(\alpha+4)}$,
 $B_2(n) \cong \frac{Nn^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+4)}$,
- c) if $M > 0$ and $N = 0$ then $B_0(n) = 1$, $B_1(n) \cong \frac{-Mn^\alpha}{\Gamma(\alpha+2)}$, $B_2(n) = 0$.

If we denote $\{\hat{L}_n^{(\alpha, M, N)}(x)\}_{n \geq 0}$ the sequence of polynomials orthonormal with respect to the inner product (7), i.e.,

$$\begin{aligned} \delta_{n,m} &= \int_0^\infty \hat{L}_n^{(\alpha, M, N)}(x) \hat{L}_m^{(\alpha, M, N)}(x) d\mu(x) \\ &\quad + M \hat{L}_n^{(\alpha, M, N)}(0) \hat{L}_m^{(\alpha, M, N)}(0) + N (\hat{L}_n^{(\alpha, M, N)})'(0) (\hat{L}_m^{(\alpha, M, N)})'(0), \end{aligned}$$

then

$$\hat{L}_n^{(\alpha, M, N)}(x) = \lambda_n L_n^{(\alpha, M, N)}(x)$$

where, taking into account (4), for $n \geq 2$ we get

$$\begin{aligned} \lambda_n^{-2} &= \langle L_n^{(\alpha, M, N)}(x), L_n^{(\alpha, M, N)}(x) \rangle \\ &= [B_0(n) - nB_1(n) + n(n-1)B_2(n)] \langle L_n^{(\alpha, M, N)}(x), (-1)^n x^n / n! \rangle \\ &= \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} [B_0(n) - nB_1(n) + n(n-1)B_2(n)] \\ &\quad [B_0(n) - (n + \alpha + 1)B_1(n) + (n + \alpha + 2)(n + \alpha + 1)B_2(n)]. \end{aligned}$$

As a conclusion, using the estimates for $B_0(n)$, $B_1(n)$, and $B_2(n)$ given above, we deduce

Proposition 1. *Let $\alpha > -1$. Then*

$$\lambda_n \cong \sqrt{\Gamma(\alpha + 1)} \begin{cases} \frac{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}{MN} n^{-5\alpha/2-4} & \text{if } M > 0, N > 0, \\ \frac{(\alpha+1)\Gamma(\alpha+4)}{N(\alpha+2)} n^{-3\alpha/2-3} & \text{if } M = 0, N > 0, \\ \frac{\Gamma(\alpha+2)}{M} n^{-3\alpha/2-1} & \text{if } M > 0, N = 0. \end{cases}$$

Proposition 2. *The representation of the polynomials $\hat{L}_n^{(\alpha, M, N)}$ in terms of the $\hat{L}_n^{(\alpha)}$ is*

$$\hat{L}_n^{(\alpha, M, N)}(x) = b_0(n) \hat{L}_n^{(\alpha)}(x) + b_1(n) x \hat{L}_{n-1}^{(\alpha+2)}(x) + b_2(n) x^2 \hat{L}_{n-2}^{(\alpha+4)}(x)$$

where

a) if $M > 0$ and $N > 0$ then

$$\begin{aligned} b_0(n) &= B_0(n) \lambda_n \{h_n^{(\alpha)}\}^{1/2} \cong -\frac{\Gamma(\alpha + 3)}{M} n^{-\alpha-1}, \\ b_1(n) &= B_1(n) \lambda_n \{h_{n-1}^{(\alpha+2)}\}^{1/2} \cong -\frac{(\alpha + 2) \sqrt{\Gamma(\alpha + 1) \Gamma(\alpha + 3)}}{M} n^{-\alpha-1}, \end{aligned}$$

$$b_2(n) = B_2(n)\lambda_n\{h_{n-2}^{(\alpha+4)}\}^{1/2} \cong \sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+5)}},$$

b) if $M = 0$ and $N > 0$ then

$$b_0(n) \cong -\frac{1}{\alpha+2},$$

$$b_1(n) \cong -\sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3)}},$$

$$b_2(n) \cong \frac{1}{\alpha+2}\sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+5)}},$$

c) if $M > 0$ and $N = 0$ then

$$b_0(n) \cong \frac{\Gamma(\alpha+2)}{M}n^{-\alpha-1},$$

$$b_1(n) \cong -\sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3)}},$$

$$b_2(n) = 0.$$

Taking into account Proposition 1 and [2, Theorem 2, (b)] we deduce a Mehler-Heine type formula for orthonormal Laguerre-Sobolev polynomials. Notice that this formula can also be proved using (5) and Proposition 2.

Proposition 3. *If we denote*

$$g_i(x) = x^{-\alpha/2}J_{\alpha+2i}(2\sqrt{x}),$$

then

$$\lim_{n \rightarrow \infty} \frac{\hat{L}_n^{(\alpha, M, N)}(x/(n+j))}{n^{\alpha/2}} = \sqrt{\Gamma(\alpha+1)} \begin{cases} g_2(x) & \text{if } M > 0, N > 0, \\ \frac{1}{\alpha+2}g_2(x) - g_1(x) - \frac{1}{\alpha+2}g_0(x) & \text{if } M = 0, N > 0, \\ -g_1(x) & \text{if } M > 0, N = 0. \end{cases}$$

uniformly on compact subsets of \mathbf{C} and uniformly on $j \in \mathbf{N} \cup \{0\}$.

Proposition 4. *The polynomials $\hat{L}_n^{(\alpha, M, N)}$ satisfy the following estimate*

$$|e^{-x/2} \hat{L}_n^{(\alpha, M, N)}(x)| \leq cn^{\alpha/2} d(x, \nu)$$

on $[0, +\infty)$, where

$$d(x, \nu) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/\nu, \\ (x\nu)^{-\alpha/2-1/4} & \text{if } 1/\nu \leq x \leq \nu/2, \\ (x\nu)^{-\alpha/2} (\nu(\nu^{1/3} + |x - \nu|))^{-1/4} & \text{if } \nu/2 \leq x \leq 3\nu/2, \\ (x\nu)^{-\alpha/2} e^{-cx} & \text{if } 3\nu/2 \leq x, \end{cases}$$

$\nu = 4n + 2\alpha + 2$, and c is a positive constant.

Proof. It is known that the polynomials $\hat{L}_n^{(\alpha)}$ satisfy the estimate (see [4], [13])

$$|e^{-x/2} \hat{L}_n^{(\alpha)}(x)| \leq cn^{\alpha/2} d(x, \nu).$$

With n, α replaced by $n - i$ and $\alpha + 2i$, $i = 0, 1, 2$, respectively, we obtain $4(n - i) + 2(\alpha + 2i) + 2 = \nu$ and

$$\begin{aligned} & |e^{-x/2} \hat{L}_{n-i}^{(\alpha+2i)}(x)| \\ & \leq cn^{(\alpha+2i)/2} \begin{cases} 1 & \text{if } 0 \leq x \leq 1/\nu, \\ (x\nu)^{-i} (x\nu)^{-\alpha/2-1/4} & \text{if } 1/\nu \leq x \leq \nu/2, \\ (x\nu)^{-i} (x\nu)^{-\alpha/2} (\nu(\nu^{1/3} + |x - \nu|))^{-1/4} & \text{if } \nu/2 \leq x \leq 3\nu/2, \\ (x\nu)^{-i} (x\nu)^{-\alpha/2} e^{-cx} & \text{if } 3\nu/2 \leq x \end{cases} \\ & = cn^{\alpha/2} \begin{cases} n^i & \text{if } 0 \leq x \leq 1/\nu, \\ x^{-i} (n/\nu)^i (x\nu)^{-\alpha/2-1/4} & \text{if } 1/\nu \leq x \leq \nu/2, \\ x^{-i} (n/\nu)^i (x\nu)^{-\alpha/2} (\nu(\nu^{1/3} + |x - \nu|))^{-1/4} & \text{if } \nu/2 \leq x \leq 3\nu/2, \\ x^{-i} (n/\nu)^i (x\nu)^{-\alpha/2} e^{-cx} & \text{if } 3\nu/2 \leq x. \end{cases} \end{aligned}$$

Thus

$$|e^{-x/2}x^i\hat{L}_{n-i}^{(\alpha+2i)}(x)| \leq cn^{\alpha/2} \begin{cases} (xn)^i & \text{if } 0 \leq x \leq 1/\nu, \\ (n/\nu)^i(x\nu)^{-\alpha/2-1/4} & \text{if } 1/\nu \leq x \leq \nu/2, \\ (n/\nu)^i(x\nu)^{-\alpha/2}(\nu(\nu^{1/3}+|x-\nu|))^{-1/4} & \text{if } \nu/2 \leq x \leq 3\nu/2, \\ (n/\nu)^i(x\nu)^{-\alpha/2}e^{-cx} & \text{if } 3\nu/2 \leq x. \end{cases}$$

Taking into account $\nu \sim n$

$$|e^{-x/2}x\hat{L}_{n-1}^{(\alpha+2)}(x)| \leq cn^{\alpha/2}d(x, \nu)$$

and

$$|e^{-x/2}x^2\hat{L}_{n-2}^{(\alpha+4)}(x)| \leq cn^{\alpha/2}d(x, \nu).$$

From these inequalities and Proposition 2, the statement follows. \square

Proposition 5. *Let $M, N \geq 0$. For $\alpha > -1/2$*

$$(9) \quad \left(\int_0^\infty |\hat{L}_n^{(\alpha, M, N)}(x)e^{-x/2}|^p x^\alpha dx \right)^{1/p} \geq \begin{cases} cn^{-1/4}(\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ cn^{\alpha/2-(\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases}$$

and for $\alpha > -2/p$

$$(10) \quad \left(\int_0^\infty |\hat{L}_n^{(\alpha, M, N)}(x)e^{-x/2}x^{\alpha/2}|^p dx \right)^{1/p} \geq \begin{cases} cn^{-1/4}(\log n)^{1/p} & \text{if } p = 4, \\ cn^{-1/p} & \text{if } 4 < p < \infty. \end{cases}$$

Proof. According to Proposition 3 and (5)

$$\begin{aligned} \int_0^\infty |\hat{L}_n^{(\alpha, M, N)}(x)e^{-x/2}|^p x^\beta dx &> \int_0^{1/\sqrt{n}} |\hat{L}_n^{(\alpha, M, N)}(x)e^{-x/2}|^p x^\beta dx \\ &\geq cn^{-\beta-1} \int_0^{\sqrt{n}} |\hat{L}_n^{(\alpha, M, N)}(t/n)|^p t^\beta dt \cong cn^{p\alpha/2-\beta-1} \\ &\times \int_0^{\sqrt{n}} t^\beta |c_2 g_2(t) - c_1 g_1(t) - c_0 g_0(t)|^p dt \geq cn^{p\alpha/2-\beta-1} \\ &\times \int_0^{2\sqrt[4]{n}} t^{2\beta-p\alpha+1} |c_2 J_{\alpha+4}(t) - c_1 J_{\alpha+2}(t) - c_0 J_\alpha(t)|^p dt. \end{aligned}$$

From the Stempak's lemma (see [16, Lemma 2.1]), for $\alpha > -1$, $\gamma > -1 - p\alpha$, and $1 \leq p < \infty$ we have

$$\int_0^m t^\gamma |c_2 J_{\alpha+4}(t) - c_1 J_{\alpha+2}(t) - c_0 J_\alpha(t)|^p dt \geq \begin{cases} c & \text{if } \gamma < p/2 - 1, \\ c \log m & \text{if } \gamma = p/2 - 1. \end{cases}$$

Let $m \in \mathbf{N}$ be such that $m \leq 2\sqrt[4]{n} < m+1$. For $\beta = \alpha$ relation (10) follows and for $\beta = p\alpha/2$ we get (11). \square

Remark 1. In particular, for $M = N = 0$, Proposition 5 extends the lower estimate to negative α in [13, Lemma 1] and [14, Lemma 1].

Now from Lemma 1, Proposition 2, and Proposition 5 we get

Corollary 1. Let $\alpha \geq 0$ and $M, N \geq 0$. Then

$$\left(\int_0^\infty |\hat{L}_n^{(\alpha, M, N)}(x) e^{-x/2}|^p x^\alpha dx \right)^{1/p} \sim \begin{cases} n^{-1/4} (\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha/2 - (\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p \leq \infty, \end{cases}$$

and

$$\left(\int_0^\infty |\hat{L}_n^{(\alpha, M, N)}(x) e^{-x/2} x^{\alpha/2}|^p dx \right)^{1/p} \sim \begin{cases} n^{-1/4} (\log n)^{1/p} & \text{if } p = 4, \\ n^{-1/p} & \text{if } 4 < p \leq \infty. \end{cases}$$

4. A COHEN TYPE INEQUALITY FOR LAGUERRE-SOBOLEV EXPANSIONS

First we will introduce some notation. We shall say that $f \in L^p(x^\beta dx)$ if f is measurable on $(0, \infty)$ and $\|f\|_{L^p(x^\beta dx)} < \infty$, where

$$\|f\|_{L^p(x^\beta dx)} = \begin{cases} \left(\int_0^\infty |f(x) e^{-x/2}|^p x^\beta dx \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{0 < x < \infty} |f(x) e^{-x/2}| & \text{if } p = \infty, \beta = \alpha, \\ \operatorname{ess\,sup}_{0 < x < \infty} |f(x) e^{-x/2} x^{\alpha/2}| & \text{if } p = \infty, \beta = p\alpha/2. \end{cases}$$

When either $\beta = \alpha$ or $\beta = p\alpha/2$ we will use the notation $L^p(x^\alpha dx)$ and $L^p(dx)$, respectively.

Now let introduce the Sobolev-type spaces

$$(11) \quad S_p^\beta = \{f : \|f\|_{S_p^\beta}^p = \|f\|_{L^p(x^\beta dx)}^p + M|f(0)|^p + N|f'(0)|^p < \infty\}, \quad 1 \leq p < \infty,$$

$$S_\infty^\beta = \{f : \|f\|_{S_\infty^\beta} = \|f\|_{L^\infty(x^\beta dx)} < \infty\}, \quad p = \infty.$$

Throughout this paper we denote by $[S_p^\beta]$ the space of all bounded linear operators $T : S_p^\beta \rightarrow S_p^\beta$, with the usual operator norm

$$\|T\|_{[S_p^\beta]} = \sup_{0 \neq f \in S_p^\beta} \frac{\|T(f)\|_{S_p^\beta}}{\|f\|_{S_p^\beta}}.$$

For $f \in S_1^\beta$, the Fourier expansion in terms of Laguerre-Sobolev type polynomials is

$$(12) \quad \sum_{k=0}^{\infty} \hat{f}(k) \hat{L}_k^{(\alpha, M, N)}(x),$$

where

$$\hat{f}(k) = \langle f, \hat{L}_k^{(\alpha, M, N)} \rangle, \quad k = 0, 1, \dots$$

The Cesàro means of order δ of the expansion (13) is defined by (see [18, pp. 76-77])

$$\sigma_n^\delta f(x) = \sum_{k=0}^n \frac{A_{n-k}^\delta}{A_n^\delta} \hat{f}(k) \hat{L}_k^{(\alpha, M, N)}(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$.

For a function $f \in S_p$ and a family of complex numbers $\{c_{k,n}\}_{k=0}^n$, $n \in N \cup \{0\}$, with $|c_{n,n}| > 0$ we define the operators $T_n^{\alpha, M, N}$ by

$$T_n^{\alpha, M, N}(f) = \sum_{k=0}^n c_{k,n} \hat{f}(k) \hat{L}_k^{(\alpha, M, N)}.$$

Let us denote $q_0 = \frac{4\alpha+4}{2\alpha+1}$ for $\beta = \alpha$ and $q_0 = 4$ for $\beta = p\alpha/2$, and let p_0 be the conjugate of q_0 . Now we can state our main result.

Theorem 1. *Let $M, N \geq 0$ and $1 \leq p \leq \infty$.*

For $\alpha > -1/2$

$$\|T_n^{\alpha, M, N}\|_{[S_p^\alpha]} \geq c|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}} & \text{if } a \leq p < p_0, \\ (\log n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0, \quad p = q_0, \\ n^{\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}} & \text{if } q_0 < p \leq b. \end{cases}$$

For $\alpha > -2/p$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$

$$\|T_n^{\alpha, M, N}\|_{[S_p^{\alpha/2}]} \geq c|c_{n,n}| \begin{cases} n^{\frac{2}{p}-\frac{3}{2}} & \text{if } a \leq p < p_0, \\ (\log n)^{\frac{1}{4}} & \text{if } p = p_0, \ p = q_0, \\ n^{\frac{1}{2}-\frac{2}{p}} & \text{if } q_0 < p \leq b, \end{cases}$$

where

- i) If $M = 0$, $N \geq 0$, then $a = 1$ and $b = \infty$,
- ii) If $M > 0$, $N \geq 0$, then $a > 1$, $b < \infty$, and $1/a + 1/b = 1$.

Remark 2. In particular, for $M = N = 0$, Theorem 1 constitutes an extension of the statement of Theorem 1 in [14] to negative values of α bigger than -1 .

Corollary 2. Let β , p_0 , q_0 , and p be as in Theorem 1. For $c_{k,n} = 1$, $k = 0, \dots, n$, and for p outside the Pollard interval (p_0, q_0) we get

$$\|S_n\|_{[S_p^\beta]} \rightarrow \infty, \quad n \rightarrow \infty,$$

where S_n denotes the n th partial sum of the expansion (13).

For $c_{k,n} = \frac{A_{n-k}^\delta}{A_n^\delta}$, $0 \leq k \leq n$, from Theorem 1 we get

Corollary 3. Let $M, N \geq 0$ and $1 \leq p \leq \infty$.

For $\alpha > -1/2$,

$$\begin{cases} 0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2} & \text{if } a \leq p < p_0, \\ 0 \leq \delta < \frac{2\alpha+1}{2} - \frac{2\alpha+2}{p} & \text{if } q_0 < p \leq b, \end{cases}$$

and $p \notin [p_0, q_0]$

$$\|\sigma_n^\delta\|_{[S_p^\alpha]} \rightarrow \infty, \quad n \rightarrow \infty.$$

For $\alpha > -2/p$ if $1 \leq p < \infty$ and $\alpha \geq 0$ if $p = \infty$,

$$\begin{cases} 0 \leq \delta < \frac{2}{p} - \frac{3}{2} & \text{if } a \leq p < p_0, \\ 0 \leq \delta < \frac{1}{2} - \frac{2}{p} & \text{if } q_0 < p \leq b, \end{cases}$$

and $p \notin [p_0, q_0]$

$$\|\sigma_n^\delta\|_{[S_p^{\alpha/2}]} \rightarrow \infty, \quad n \rightarrow \infty.$$

For the proof of Theorem 1 we will use the test functions

$$g_n^{\alpha,j}(x) = n^{-\alpha/2} \left[x^j L_n^{(\alpha+j)}(x) - \sqrt{\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)}} x^j L_{n+2}^{(\alpha+j)}(x) \right],$$

where $j \in \mathbf{N} \setminus \{1\}$. Notice that $g_n^{\alpha,j}(0) = 0$ and $(g_n^{\alpha,j})'(0) = 0$.

This function can be written as (see [14, formula (2.15)])

$$g_n^{\alpha,j}(x) = n^{-\alpha/2} \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) L_{n+m}^{(\alpha)}(x),$$

where

$$a_{0,j}(\alpha, n) = \frac{\Gamma(n+\alpha+j+1)}{\Gamma(n+\alpha+1)} \cong n^j,$$

$$a_{j+2,j}(\alpha, n) = (-1)^{j+1} \sqrt{\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)}} \times \frac{\Gamma(n+j+3)}{\Gamma(n+3)} \cong (-1)^{j+1} n^j.$$

Applying the operator $T_n^{\alpha,M,N}$ to the test functions $g_n^{\alpha,j}$, for some $j > \alpha - 1/2 - (2\alpha + 2)/p$, we get

$$(13) \quad T_n^{\alpha,M,N}(g_n^{\alpha,j}) = \sum_{k=0}^n c_{k,n}(g_n^{\alpha,j})^{\wedge}(k) \hat{L}_k^{(\alpha,M,N)},$$

where

$$(g_n^{\alpha,j})^{\wedge}(k) = \langle g_n^{\alpha,j}, \hat{L}_k^{(\alpha,M,N)} \rangle, \quad k = 0, 1, \dots, n.$$

From Proposition 2

$$\begin{aligned} \Gamma(\alpha+1)(g_n^{\alpha,j})^{\wedge}(k) &= n^{-\alpha/2} \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^{(\alpha)}(x) \hat{L}_k^{(\alpha,M,N)} x^\alpha e^{-x} dx \\ &= n^{-\alpha/2} b_0(k) \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^{(\alpha)}(x) \hat{L}_k^{(\alpha)} x^\alpha e^{-x} dx \\ &\quad + n^{-\alpha/2} b_1(k) \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^{(\alpha)}(x) x \hat{L}_{k-1}^{(\alpha+2)} x^\alpha e^{-x} dx \\ &\quad + n^{-\alpha/2} b_2(k) \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^{(\alpha)}(x) x^2 \hat{L}_{k-2}^{(\alpha+4)} x^\alpha e^{-x} dx \\ &= I_1(k, n) + I_2(k, n) + I_3(k, n), \end{aligned}$$

where $0 \leq k \leq n$ and, as a convention, $\hat{L}_i^{(\alpha)}(x) = 0$, for $i = -1, -2$.

According to (1)

$$I_1(k, n) = n^{-\alpha/2} b_0(k) (h_k^{(\alpha)})^{-1/2} \sum_{m=0}^{j+2} a_{m,j}(\alpha, n) \int_0^\infty L_{n+m}^{(\alpha)}(x) L_k^{(\alpha)} x^\alpha e^{-x} dx.$$

Thus, for $0 \leq k \leq n-1$

$$I_1(k, n) = 0.$$

Let $n \geq 0$. Then

$$I_1(n, n) = n^{-\alpha/2} b_0(n) (h_n^{(\alpha)})^{-1/2} a_{0,j}(\alpha, n) \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \cong n^j \sqrt{\Gamma(\alpha+1)} b_0(n).$$

In a similar way, for $0 \leq k \leq n-1$

$$I_2(k, n) = 0,$$

and, for $n \geq 1$ from (2) and (3)

$$\begin{aligned} I_2(n, n) &= -n^{1-\alpha/2} \sqrt{\frac{\Gamma(\alpha+3)\Gamma(n)}{\Gamma(n+\alpha+2)}} b_1(n) a_{0,j}(\alpha, n) \\ &\quad \times \int_0^\infty L_n^{(\alpha)}(x) L_n^{(\alpha+1)} x^\alpha e^{-x} dx \cong -\sqrt{\Gamma(\alpha+3)} b_1(n) n^j. \end{aligned}$$

Finally, for $0 \leq k \leq n-1$

$$I_2(k, n) = 0,$$

and from (9) and (3) for $n \geq 2$ we get

$$\begin{aligned} I_3(n, n) &= n(n-1) n^{-\alpha/2} \sqrt{\frac{\Gamma(\alpha+5)\Gamma(n-1)}{\Gamma(n+\alpha+3)}} b_2(n) a_{0,j}(\alpha, n) \\ &\quad \times \int_0^\infty L_n^{(\alpha)}(x) L_n^{(\alpha+2)} x^\alpha e^{-x} dx \cong \sqrt{\Gamma(\alpha+5)} b_2(n) n^j. \end{aligned}$$

In order to estimate $(g_n^{\alpha,j})^\wedge(k)$, we will analyze the following three situations.

1. If $M > 0$ and $N > 0$, then

$$\begin{aligned} I_1(n, n) &\cong -\frac{\Gamma(\alpha+3)\sqrt{\Gamma(\alpha+1)}}{M} n^{j-\alpha-1}, \\ I_2(n, n) &\cong -\frac{(\alpha+2)\Gamma(\alpha+3)\sqrt{\Gamma(\alpha+1)}}{M} n^{j-\alpha-1}, \\ I_3(n, n) &\cong \sqrt{\Gamma(\alpha+1)} n^j. \end{aligned}$$

Thus

$$(g_n^{\alpha,j})^\wedge(n) = I_1(n, n) + I_2(n, n) + I_3(n, n) \cong \sqrt{\Gamma(\alpha+1)} n^j.$$

2. If $M = 0$ and $N > 0$, then

$$\begin{aligned} I_1(n, n) &\cong -\frac{\sqrt{\Gamma(\alpha+1)}n^j}{\alpha+1}, \\ I_2(n, n) &\cong \sqrt{\Gamma(\alpha+1)}n^j, \\ I_3(n, n) &\cong \frac{\sqrt{\Gamma(\alpha+1)}n^j}{\alpha+1}. \end{aligned}$$

Thus

$$(g_n^{\alpha,j})^\wedge(n) \cong \sqrt{\Gamma(\alpha+1)}n^j.$$

3. If $M > 0$ and $N = 0$, then

$$\begin{aligned} I_1^{n,n} &\cong \frac{\Gamma(\alpha+2)\sqrt{\Gamma(\alpha+1)}}{M}n^{j-\alpha-1}, \\ I_2^{n,n} &\cong \sqrt{\Gamma(\alpha+1)}n^j, \\ I_3^{n,n} &= 0. \end{aligned}$$

Thus

$$(g_n^{\alpha,j})^\wedge(n) \cong \sqrt{\Gamma(\alpha+1)}n^j.$$

As a conclusion,

$$(14) \quad \begin{cases} (g_n^{\alpha,j})^\wedge(k) = 0, & 0 \leq k \leq n-1, \\ (g_n^{\alpha,j})^\wedge(n) \cong \sqrt{\Gamma(\alpha+1)}n^j. \end{cases}$$

On the other hand, from [14, formula (3.3)]

$$(15) \quad \|g_n^{\alpha,j}\|_{S_p^\alpha} = \|g_n^{\alpha,j}\|_{L^p(x^\alpha dx)} \leq cn^{j-\alpha/2-1/2+(\alpha+1)/p},$$

and from [14, formula (1.19), (2.12)]

$$(16) \quad \|g_n^{\alpha,j}\|_{S_p^{p\alpha/2}} = \|g_n^{\alpha,j}\|_{L^p(dx)} \leq cn^{-\alpha/2}n^{(\alpha+j)/2}n^{j/2-1/2+1/p} = cn^{j-1/2+1/p}.$$

With this background we can prove our main result

Proof of Theorem 1. By duality, it is enough to assume that $q_0 \leq p \leq b$. From (14), (15), (16), and (17)

$$\begin{aligned} (17) \quad \|T_n^{\alpha,M,N}\|_{S_p^\beta} &\geq [\|g_n^{(\alpha,j)}\|_{S_p^\beta}]^{-1} \|T_n^{\alpha,M,N} g_n^{(\alpha,j)}\|_{S_p^\beta} \\ &\geq c|c_{n,n}| n^j \begin{cases} n^{-j+\alpha/2+1/2-(\alpha+1)/p} \|\hat{L}_n^{(\alpha,M,N)}\|_{S_p^\alpha}, \\ n^{-j+1/2-1/p} \|\hat{L}_n^{(\alpha,M,N)}\|_{S_p^{p\alpha/2}}. \end{cases} \end{aligned}$$

From (12) and Proposition 5 we obtain

$$(18) \quad \|\hat{L}_n^{(\alpha, M, N)}\|_{S_p^\alpha} \geq \begin{cases} cn^{-1/4}(\log n)^{1/p} & \text{if } p = q_0, \\ cn^{\alpha/2 - (\alpha+1)/p} & \text{if } q_0 < p < \infty, \end{cases}$$

and

$$(19) \quad \|\hat{L}_n^{(\alpha, M, N)}\|_{S_p^{\alpha/2}} \geq c \begin{cases} cn^{-1/4}(\log n)^{1/p} & \text{if } p = q_0, \\ cn^{-1/p} & \text{if } q_0 < p < \infty. \end{cases}$$

On the other hand,

$$(20) \quad |\hat{L}_n^{(\alpha, 0, 0)}(0)| \sim n^{\alpha/2} \sim |\hat{L}_n^{(\alpha, 0, N)}(0)|.$$

Now from (18), (19), (20), and (21) the statement of the theorem follows.

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